Hölder Inequalities and QCD Sum-Rule Bounds on the Masses of Light Quarks

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QCD Laplace Sum-Rules must satisfy a fundamental Hölder inequality if they are to consistently represent an integrated hadronic spectral function. The Laplace sum-rules of pion currents is shown to violate this inequality unless the u and d quark masses are sufficiently large, placing a lower bound on $m_u + m_d$, the $SU(2)$-invariant combination of the light-quark masses.

In this paper we briefly review the development of Hölder inequalities for QCD sum-rules[1] and their application to obtain light-quark (u, d) mass bounds.[2]

Laplace sum-rules for pseudoscalar currents with quantum numbers of the pion relate a QCD prediction $R_5\left(M^2\right)$ to the integral of the associated hadronic spectral function $\rho_5(t)$

$$R_5\left(M^2\right) = \frac{1}{\pi}\int\limits_{t_0}^{\infty} \rho(t)\exp\left(-\frac{t}{M^2}\right)dt \quad , \tag{1}$$

where t_0 is the physical threshold for the spectral function. Since $\rho_5(t) \geq 0$, the right-hand (phenomenological side) side of (1) must satisfy integral inequalities over a measure $d\mu = \rho_5(t)\,dt$.

Hölder's inequality over a measure $d\mu$ is

$$\left|\int_{t_1}^{t_2} f(t)g(t)d\mu\right| \leq \left(\int_{t_1}^{t_2}\left|f(t)\right|^p d\mu\right)^{\frac{1}{p}}\left(\int_{t_1}^{t_2}\left|g(t)\right|^q d\mu\right)^{\frac{1}{q}} \ ,\ \frac{1}{p}+\frac{1}{q}=1\ ;\ p,\ q\geq 1 \quad , \tag{2}$$

which for $p = q = 2$ reduces to the familiar Schwarz inequality, implying that the Hölder inequality is a more general constraint. The Hölder inequality can be applied to Laplace sum-rules by identifying $d\mu = \rho(t)\,dt$, $\tau = 1/M^2$ and defining

$$S_5\left(\tau\right) = \frac{1}{\pi}\int\limits_{\mu_{th}}^{\infty}\rho_5(t)e^{-t\tau}\,dt \tag{3}$$

where μ_{th} will later be identified as lying above m_π^2. Suitable choices of $f(t)$ and $g(t)$ in the Hölder inequality (2) yield the following inequality for $S_5(t)$:[1]

$$S_5\left(\tau + (1-\omega)\delta\tau\right) \leq S_5^{\omega}\left(\tau\right) S_5^{1-\omega}\left(\tau+\delta\tau\right) \quad , \ \forall\ 0 \leq \omega \leq 1 \quad . \tag{4}$$

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The Laplace sum-rule relating QCD and hadronic physics is obtained by applying the Borel transform operator[3] $\hat{B}$ to the dispersion relation

$$\Pi_5\left(Q^2\right)=a+bQ^2+\frac{Q^4}{\pi}\int_{t_0}^{\infty}\frac{\rho_5(t)}{t^2\left(t+Q^2\right)}\,dt\quad . \tag{5}$$

The quantity $\Pi_5\left(Q^2\right)$ is the QCD prediction for the correlation function of pseudoscalar (pion) currents

$$\Pi_5\left(Q^2\right)=i\int d^4x\,e^{iq\cdot x}\,\langle O|T\left[J_5(x)J_5(0)\right]|O\rangle \tag{6}$$

$$J_5(x)=\frac{1}{\sqrt{2}}\left(m_u+m_d\right)\left[\bar{u}(x)i\gamma_5u(x)-\bar{d}(x)i\gamma_5d(x)\right]\quad , \tag{7}$$

and the theoretically-determined quantity $R_5\left(M^2\right)$ is obtained from the Borel transform of the QCD correlation function.

$$R_5\left(M^2\right)=M^2\hat{B}\left[\Pi_5\left(Q^2\right)\right] \tag{8}$$

Perturbative contributions to $R_5\left(M^2\right)$ are known up to four-loop order.[4,5] Infinite correlation-length vacuum effects in $R_5\left(M^2\right)$ are represented by the (non-perturbative) QCD condensate contributions.[3,5,6] In addition to the QCD condensate contributions the pseudoscalar (and scalar) correlation functions are sensitive to finite correlation-length vacuum effects described by direct instantons in the instanton liquid model.[7] The total result for $R_5\left(M^2\right)$ to leading order in the light-quark masses is[2]

$$\begin{aligned}R_5\left(M^2\right)=&\frac{3m^2M^4}{8\pi^2}\left(1+4.821098\frac{\alpha}{\pi}+21.97646\left(\frac{\alpha}{\pi}\right)^2+53.14179\left(\frac{\alpha}{\pi}\right)^3\right)\\&+m^2\left(-\langle m\bar{q}q\rangle+\frac{1}{8\pi}\langle\alpha G^2\rangle+\frac{\pi\langle\mathcal{O}_6\rangle}{4M^2}\right)\\&+m^2\frac{3\rho_c^2M^6}{8\pi^2}e^{-\rho_c^2M^2/2}\left[K_0\left(\rho_c^2M^2/2\right)+K_1\left(\rho_c^2M^2/2\right)\right]\quad ,\end{aligned} \tag{9}$$

where α and $m=\left(m_u+m_d\right)/2$ are the $\overline{MS}$ running coupling and quark masses at the scale M, and ρ_c represents the instanton size in the instanton liquid model. Note that all the theoretical contributions are proportional to m^2, demonstrating that the quark mass sets the scale of the pseudoscalar channel. Higher-loop perturbative contributions in (9) are significant, and effectively enhance the quark mass with increasing loop order.

To employ the Hölder inequality (4) we separate out the pion pole by setting $\mu_{th}=9m_\pi^2$ in (3).

$$S_5\left(M^2\right)=R_5\left(M^2\right)-2f_\pi^2m_\pi^4=\int_{9m_\pi^2}^{\infty}\rho_5(t)e^{-t\tau}\,dt \tag{10}$$

Lower bounds on the quark mass m can then be obtained by finding the minimum value of m for which the Hölder inequality (4) is satisfied. Introducing further phenomenological contributions (*e.g.* three-pion continuum) would tend to give a larger mass bound. However, if only the pion pole is separated out, then the analysis is not subject to uncertainties introduced by the phenomenological model.

Standard values of the QCD parameters are employed in the inequality analysis of (4), and we use $\delta\tau \lesssim 0.1\,\mathrm{GeV}^{-2}$ for which this analysis becomes local (depending only on the Borel scale M).[1,2] Validity of QCD predictions at the τ mass is evidenced by the analysis of the τ hadronic width, hadronic contributions to $\alpha_{EM}\left(M_Z\right)$ and the muon anomalous magnetic moment,[8] so we impose the inequality (4) at the τ mass $M = M_\tau$. The resulting Hölder inequality bound on the $\overline{MS}$ quark masses scaled to $1.0\,\mathrm{GeV}$ is[2]

$$m(1\,\mathrm{GeV}) = \frac{1}{2}\left[m_u(1\,\mathrm{GeV}) + m_d(1\,\mathrm{GeV})\right] \geq 3\,\mathrm{MeV} \tag{11}$$

For comparison with other determinations of the light quark masses, this result has been converted to

$$m(2\,\mathrm{GeV}) = \frac{1}{2}\left[m_u(2\,\mathrm{GeV}) + m_d(2\,\mathrm{GeV})\right] \geq 2.1\,\mathrm{MeV} \tag{12}$$

by the Particle Data Group.[9]

The theoretical uncertainties in the quark mass bound (11) from the QCD parameters and (estimated) higher-order perturbative effects are less than 5%, and the result (11) is the absolute lowest bound resulting from the uncertainty analysis.[2] for $M \gtrsim M_\tau$ the theoretical uncertainties in the mass bound are $\lesssim 0.1\,\mathrm{MeV}$. Thus we have not extracted mass bounds below the energy scale $M \approx M_\tau$ at which theoretical uncertainties first reach a non-negligible level. Finally, compared with the positivity inequality $S_5\left(M^2\right) \geq 0$ (as first used to obtain quark mass bounds from QCD sum-rules[5]) the Hölder inequality leads to quark mass bounds larger by a factor of 2 for identical theoretical and phenomenological inputs at $M = M_\tau$, demonstrating that the Hölder inequality provides stringent constraints on the quark mass.

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